JOURNAL OF COMPUTATIONAL PHYSICS **87**, 301–327 (1990)

Numerical Computation of the Incomplete Lipschitz–Hankel Integral $Je_0(a, z)$

STEVEN L. DVORAK[1] AND EDWARD F. KUESTER

Electromagnetics Laboratory, Department of Electrical and Computer Engineering, Campus Box 425, University of Colorado, Boulder, Colorado 80309

Received December 6, 1988; revised May 1, 1989

Two factorial-Neumann series expansions are derived for the incomplete Lipschitz–Hankel integral $Je_0(a, z)$. These expansions are used together with the Neumann series expansion, given by Agrest, in an algorithm which efficiently computes $Je_0(a, z)$ to a user defined number of significant digits. Other expansions for $Je_0(a, z)$, which are found in the literature, are also discussed, but these expansions are found to offer no significant computational advantages when compared with the expansions used in the algorithm.

1. INTRODUCTION

The incomplete Lipschitz–Hankel integral (ILHI) is defined in [1] as

$$Ze_\nu(a, z) := \int_0^z e^{-at} t^\nu Z_\nu(t)\, dt; \qquad a, z, \nu \in \mathbb{C}, \tag{1}$$

where $Z_\nu(t)$ is one of the cylindrical functions $J_\nu(t)$, $Y_\nu(t)$, $I_\nu(t)$, or $K_\nu(t)$. For finite values of z, this integral will converge provided $\Re(2\nu + 1) > 0$. The corresponding complete (ordinary) Lipschitz–Hankel integral is an improper integral of the form

$$\int_0^\infty e^{-at} t^{\mu-1} Z_\nu(t)\, dt; \qquad a, \nu, \mu \in \mathbb{C}, \tag{2}$$

where the conditions $\Re(\nu + \mu) > 0$ and $\Re(a) > 0$ guarantee its convergence.

ILHIs are important special functions since they arise in a number of problems in mathematical physics. Agrest and Maksimov [1] describe a number of these and provide reference to a large body of literature on the subject. A typical example from acoustics, which involves the functions $Je_0(a, z)$ and $Ye_0(a, z)$, is the calculation of scattering from an absorbing strip [2]. In electromagnetics, ILHIs appear in the Sommerfeld problem [3] (i.e., dipole sources above earth) as well as in the problem of a printed strip dipole antenna in a layered medium (see [4, 5]).

Several papers have been written on the computation of ILHIs. Agrest has developed various expansions (see [6–8]) which can be used to compute $Ze_0(a, z)$

[1] Present address: Department of Electrical and Computer Engineering, University of Arizona, Tucson, AZ 85721.

0021-9991/90 $3.00

in different regions of the variables a and z. Earlier, Maximon [9] obtained a Neumann series expansion for a class of functions including $Je_0(a, z)$ and $Ie_0(a, z)$, but no computational procedures were attempted. Also, Amos and Burgmeier [10] give a recurrence algorithm which can be used to compute $Je_0(a, z)$ and $Ie_0(a, z)$, although no numerical tests were reported.

In this paper, we develop an algorithm which efficiently computes $Je_0(a, z)$ to a user defined number of significant digits (SD) for $z \in \Re$ and $a \in \mathbb{C}$. We only need to handle positive values of z, since

$$Je_0(a, -z) = -Je_0(-a, z). \tag{3}$$

In Section 2, we derive a first-order nonhomogeneous recurrence relation for $Je_n(a, z)$. Then in Section 3, we use this recurrence relation for $n \geqslant 0$ to construct a factorial-Neumann series expansion for $Je_0(a, z)$ which converges rapidly for small to moderate values of $|z\sqrt{a^2+1}|$. In Section 4, the recurrence relation is now used for $n \leqslant -1$ to obtain a second factorial-Neumann series expansion. This time, we obtain an asymptotic expansion which can be used to compute $Je_0(a, z)$ for large values of $z|a^2+1|$. In Section 5, we discuss a Neumann series expansion, and some other expansions for $Je_0(a, z)$ which are found in the literature. Finally, in Section 6, the two factorial-Neumann series expansions are used in conjunction with the Neumann series expansion to develop an algorithm which efficiently computes $Je_0(a, z)$ to a user defined number of significant digits (SD). In Section 7, $Je_0(a, z)$ is computed using both the algorithm which is developed in this paper (TJE0), and an adaptive quadrature routine (D01AKF) from the NAG library [11], for some typical values of a, z, and SD. A comparison of the accuracy of the results and the required CPU time is then made for these two methods. It is found that TJE0 offers a very fast and accurate way to compute $Je_0(a, z)$.

We will restrict z to be a real number in this paper, however, the analysis which is presented in this paper can be modified for $z \in \mathbb{C}$.

2. A Recurrence Relation for $Je_n(a, z)$

The ILHI for the Bessel function of the first kind of integer order is given by (see (1))

$$Je_n(a, z) := \int_0^z e^{-at} t^n J_n(t)\, dt. \tag{4}$$

It is convenient to also define a related integral which has variable upper and lower limits

$$\mathscr{J}e_n(a, \delta, z) := \int_\delta^z e^{-at} t^n J_n(t)\, dt. \tag{5}$$

If we apply integration by parts twice to (5)—first using

$$u := e^{-at} J_n(t), \qquad dv := t^n dt, \tag{6}$$

along with the recurrence relation for the derivative of the Bessel function [12, (9.1.27)], and, second, using

$$u := e^{-at}, \qquad dv := t^{n+1} J_n(t)\, dt, \tag{7}$$

along with the indefinite integral [13, (5.52.1)]—then we obtain the recurrence relation,

$$\mathscr{J}e_n(a, \delta, z) + d_n \mathscr{J}e_{n+1}(a, \delta, z) = f_n(z) - f_n(\delta); \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{8}$$

where

$$\begin{aligned} d_n &:= -\left(\frac{a^2+1}{2n+1}\right), \\ f_n(t) &:= \frac{e^{-at} t^{n+1}}{(2n+1)} \left[J_n(t) + a J_{n+1}(t) \right]. \end{aligned} \tag{9}$$

Now, choosing δ so that $f_n(\delta) = 0$ yields the desired recurrence relation,

$$\mathscr{J}e_n(a, \delta, z) + d_n \mathscr{J}e_{n+1}(a, \delta, z) = f_n(z); \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{10}$$

The solutions of this recurrence relation behave very differently for the two cases $n \geqslant 0$ and $n \leqslant -1$. These two cases are explored in Sections 3 and 4, respectively.

3. A Convergent Factorial-Neumann Series Expansion for $Je_0(a, z)$

If we wish to use the recurrence relation (10) for $n \geqslant 0$, then we must choose δ so that $f_n(\delta) = 0$. Due to the behavior of the Bessel function for small arguments (see [12, (9.1.7)]), this requirement is satisfied by choosing $\delta = 0$. For this value of δ, reference to (4) and (5) shows that

$$Je_n(a, z) = \mathscr{J}e_n(a, 0, z). \tag{11}$$

Therefore, the recurrence relation (10) can be rewritten as

$$Je_n(a, z) + d_n Je_{n+1}(a, z) = f_n(z); \qquad n \geqslant 0. \tag{12}$$

Before we can use (12), we need to determine in which direction the recurrence will be stable. Therefore, we will use the techniques in [14] to perform a stability analysis on (12). The homogeneous solution of (12) is found by using [14, (A.24)], (9), and [12, (6.1.12)]:

$$Je_n^{(h)}(a) := \prod_{k=0}^{n-1} [-d_k]^{-1} = \left[\frac{2}{a^2+1}\right]^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)}; \qquad n \geqslant 0. \tag{13}$$

For large values of n, an approximation for (13) can be obtained by applying Stirling's formula [12, (6.1.37)] and [12, (4.1.17)]:

$$Je_n^{(h)}(a) \sim \sqrt{2}\left[\frac{2n}{e(a^2+1)}\right]^n; \qquad n \to \infty. \tag{14}$$

Now, an index of stability for the forward computation of $Je_n(a, z)$ from $Je_k(a, z)$ is given by (see [14, (2.17)])

$$\alpha(k, n) = \left|\frac{Je_k(a, z)\, Je_n^{(h)}(a)}{Je_n(a, z)\, Je_k^{(h)}(a)}\right| = \frac{\rho_n}{\rho_k}, \tag{15}$$

where

$$\rho_n := \left|\frac{Je_0(a, z)\, Je_n^{(h)}(a)}{Je_n(a, z)}\right|. \tag{16}$$

If we assume that an initial value, $Je_k(a, z)$, is known, then when recursing from k to n, where $n > k$, the error is increased if $\rho_n \geqslant \rho_k$ and decreased if $\rho_n < \rho_k$. Therefore, if we can obtain an approximation for ρ_n, then we will be able to use this index to determine whether the recurrence relation (12) can be used in the forward direction.

The behavior of ρ_n, for large values of n, is obtained by using (13) along with the asymptotic behavior of $Je_n(a, z)$ which is given in (87) in Appendix A:

$$\rho_n^{(1)} \sim |Je_0(a, z)|\, 2\sqrt{2n}\, \frac{e^{z\Re(a)}}{z}\, \Gamma\left(n+\frac{3}{2}\right)\left[\frac{4n}{ez^2|a^2+1|}\right]^n;$$
$$n \gg \kappa := \max(z, |az|), \qquad z > 0, a \in \mathbb{C}. \tag{17}$$

For the limiting case, $n \to \infty$, this expression can be simplified by once again applying Stirling's formula. This gives

$$\rho_n^{(1)} \sim |Je_0(a, z)|\, \frac{e^{z\Re(a)}}{z}\, 4n\sqrt{\pi n}\left[\frac{2n}{ez|\sqrt{a^2+1}|}\right]^{2n}; \qquad n \to \infty. \tag{18}$$

Therefore, we find that forward recurrence using (12) is unstable, since the condition (see [14, (2.18)]),

$$\lim_{n \to \infty} \rho_n = \infty, \tag{19}$$

is satisfied.

The recurrence relation (12) can also be found in [6, pp. 207]. In that paper, the authors state that in order to analyze the ILHIs with integer values of v, it is sufficient to study them with $v = 0$. It may be possible to use (12) in the forward direction to calculate a few values of $Je_n(a, z)$, but as we have shown in (18), the recurrence will eventually become unstable. The point where the forward recurrence becomes unstable is dependent on the parameters a and z.

We already know that forward recurrence is unstable when $n \gg \max(z, |az|)$, since $\rho_n^{(1)}$ is monotone increasing (see (17)). Now, we need to determine the stability when $n < z$. We will assume that $\Re(a) > 0$ for this analysis, but the case $\Re(a) \leqslant 0$ can be handled using similar techniques. We start the analysis by splitting the integral into two pieces,

$$Je_n(a, z) = Je_n(a, \infty) + \mathcal{J}e_n(a, \infty, z); \qquad z > 0, \Re(a) > 0. \tag{20}$$

The first integral is known in closed form (see [13, (6.623.1)]), and an approximation for the second integral is given in Appendix A (see (102)):

$$Je_n(a, z) \sim \frac{2^n \Gamma(n+1/2)}{\sqrt{\pi}\,(a^2+1)^{n+1/2}} - (-1)^n \sqrt{\frac{2}{\pi z}} \frac{e^{-az} z^n}{(a^2+1)} \left[a \cos\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right) - \sin\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]; \qquad z > 0, \Re(a) > 0, \eta \gg n \geqslant 0, \tag{21}$$

where $\eta := \min(z, z|a \pm j|)$. An index of stability for using (12) in the forward direction, when $0 < n < z$, is given by (15), where an approximation for ρ_n can now be obtained by substituting (13) and (21) into (16):

$$\rho_n^{(2)} \sim \left| \frac{Je_0(a, z)}{\dfrac{1}{\sqrt{a^2+1}} - \sqrt{\dfrac{2}{z}} \left[\dfrac{-z(a^2+1)}{2}\right]^n \dfrac{[a\cos(z+n\pi/2-\pi/4) - \sin(z+n\pi/2-\pi/4)]}{\Gamma(n+1/2)(a^2+1)} e^{-az}} \right|; \quad z > 0, \Re(a) > 0, \eta \gg n \geqslant 0. \tag{22}$$

Using the above approximation, we find that forward recurrence will be relatively stable when $\eta \gg n \geqslant 0$. A comparison between $\rho_n^{(1)}$ and $\rho_n^{(2)}$, given in (17) and (22), respectively, shows that forward recurrence using (12) becomes unstable somewhere in the vicinity of $n = \kappa$.

Since $\rho_n^{(1)}$ is monotone increasing as $n \to \infty$, it may be possible to calculate $Je_0(a, z)$ by using backward recurrence, in the form of a Miller algorithm. In the Miller algorithm, the initial condition,

$$Je_N(a, z) = 0, \tag{23}$$

is chosen to satisfy the asymptotic behavior given in (19). Then (12) is used in the backward direction to obtain $Je_n(a, z)$ for $n = N-1, N-2, ..., 0$. For the idealized case where infinite precision arithmetic is used, a relative error bound can be obtained by applying (see [14, p. 27])

$$\left|\frac{Je_0^N(a,z)-Je_0(a,z)}{Je_0(a,z)}\right|=\left|\frac{Je_N(a,z)}{Je_N^{(h)}(a)\,Je_0(a,z)}\right|=\frac{1}{\rho_N^{(1)}};\qquad N\geqslant\kappa,\tag{24}$$

where $\rho_N^{(1)}$ is given in (17). Reference to (17) or (18) shows that in the idealized case, Miller's algorithm can be used to compute $Je_0(a, z)$ to any desired accuracy if N is chosen large enough.

The actual error can be determined by using (see [14, (3.15)])

$$\begin{aligned}\left|\frac{Je_0^N(a,z)-Je_0(a,z)}{Je_0(a,z)}\right| &\leqslant \left|\frac{Je_0^{(h)}(a)}{Je_0(a,z)}\right|\left\{(1+\varepsilon)^N\frac{(\varepsilon+|Je_N(a,z)|)}{|Je_N^{(h)}(a)|}\right.\\ &\quad\left.+\varepsilon\sum_{k=0}^{N-1}(1+\varepsilon)^k\left|\frac{Je_k(a,z)}{Je_k^{(h)}(a)}\right|\right\}\\ &\sim(1+\varepsilon)^N\left\{\frac{\varepsilon}{|Je_0(a,z)|}\frac{\Gamma(1/2)}{\Gamma(N+1/2)}\left|\frac{a^2+1}{2}\right|^N+\frac{1}{\rho_N^{(1)}}\right\}\\ &\quad+\varepsilon\sum_{k=0}^{\kappa-1}\frac{(1+\varepsilon)^k}{\rho_k^{(2)}}+\varepsilon\sum_{k=\kappa}^{N-1}\frac{(1+\varepsilon)^k}{\rho_k^{(1)}};\\ &\qquad N\geqslant\kappa,\ z>0,\ \Re(a)>0,\end{aligned}\tag{25}$$

where the previous results, (13), (17), and (22), have been used. The ε in (25) is a measure of the maximum error introduced due to finite precision arithmetic. Equation (25) shows that errors due to finite precision arithmetic become significant, and must be included in the analysis, when either $\varepsilon/\rho_k^{(2)}$ or $\varepsilon/\rho_k^{(1)}$ become large for $0\leqslant k\leqslant\kappa-1$ or $\kappa\leqslant k\leqslant N-1$, respectively. Reference to (22) shows that $\varepsilon/\rho_k^{(2)}$ may become large when $z|a^2+1|$ is large. Therefore, one must be careful while using the Miller algorithm when this occurs. For small to moderate values of $z|a^2+1|$, the error accumulation due to finite precision arithmetic can usually be ignored and the error bound given in (24) is adequate. Errors due to finite precision arithmetic are discussed more thoroughly in [15, Section 4].

When a sequence of solutions is desired (i.e., $\{Je_n(a, z)\}$; $n=0, 1, 2, \ldots$), then a backward recurrence algorithm should be applied directly to (12); but when only one solution is desired, as in our case, the equivalent series representation,

$$Je_n^N(a,z)=Je_n^{(h)}(a)\sum_{k=n}^{N-1}\frac{f_k(z)}{Je_k^{(h)}(a)},\tag{26}$$

where $f_k(z)$ and $Je_k^{(h)}(a)$ are defined in (9) and (13), provides some computational advantages. This series representation was obtained by applying the initial condition (23) to [14, (A.25)–(A.27)] (For more details, see Appendix A of [15].) Substituting (9) and (13) into (26), yields

$$Je_n^N(a, z) = \frac{ze^{-az}}{2}\,\Gamma\left(n+\frac{1}{2}\right)\sum_{k=n}^{N-1} z^k\left[\frac{a^2+1}{2}\right]^{k-n}\frac{[J_k(z)+aJ_{k+1}(z)]}{\Gamma(k+3/2)};$$
$$n \geqslant 0,\ z>0,\ a\in\mathbb{C}. \tag{27}$$

Due to the presence of the z^k term in (27), this series is not a Neumann series. In [16], Nielsen classifies series which can be written in the form

$$\sum_{n=0}^{\infty} a_n z^{\pm n} J_{n+\nu}(z) \tag{28}$$

as *Fakultätenreihe*; we will use the English terminology factorial-Neumann series. Since (27) is of the form (28), we will call it a convergent factorial-Neumann series expansion.

In this paper, we are interested in the special case $n=0$. The desired factorial-Neumann series expansion for $Je_0(a, z)$ is given by

$$Je_0^N(a, z) = \Gamma\left(\frac{3}{2}\right) ze^{-az}\sum_{k=0}^{N-1}\left[\frac{z(a^2+1)}{2}\right]^{k}\frac{[J_k(z)+aJ_{k+1}(z)]}{\Gamma(k+3/2)};$$
$$z>0,\ a\in\mathbb{C}. \tag{29}$$

Reference to (18) and (24) shows that this series converges most rapidly for small to moderate values of $z|\sqrt{a^2+1}|$.

4. An Asymptotic Factorial-Neumann Series Expansion for $Je_0(a, z)$

Next, let us use the recurrence relation (10) for $n \leqslant -1$ to obtain a second factorial-Neumann series expansion with different properties. First, we make a change of variables, $m = -(n+1)$, in (10). Next, if we define a new integral,

$$\hat{\mathscr{J}}e_m(a, \delta, z) := \int_{\delta}^{z} e^{-at} t^{-m} J_m(t)\, dt, \tag{30}$$

then (10) can be rewritten as

$$\hat{\mathscr{J}}e_m(a, \delta, z) + \hat{d}_m \hat{\mathscr{J}}e_{m+1}(a, \delta, z) = \hat{f}_m(z); \qquad m \geqslant 0, \tag{31}$$

where

$$\hat{d}_m := \frac{1}{d_m} = -\left(\frac{2m+1}{a^2+1}\right),$$
$$\hat{f}_m(z) := \frac{e^{-az}}{(a^2+1)\, z^m}\left[J_{m+1}(z) - aJ_m(z)\right]. \tag{32}$$

Once again, we need to perform a stability analysis before (31) can be used. The homogeneous solution for (31) is obtained by applying [14, (A.24)], (13), and (14):

$$\hat{\mathscr{J}}e_m^{(h)}(a) = \prod_{k=0}^{m-1} [-\hat{d}_k]^{-1} = [Je_m^{(h)}(a)]^{-1}$$
$$= \left[\frac{a^2+1}{2}\right]^m \frac{\Gamma(1/2)}{\Gamma(m+1/2)} \tag{33}$$
$$\sim \frac{1}{\sqrt{2}}\left[\frac{e(a^2+1)}{2m}\right]^m; \qquad m \to \infty. \tag{34}$$

This time, we can use Hankel's asymptotic expansion [12, (9.2.5)] to show that the choice

$$\delta := \begin{cases} \infty; & \Re(a) \geq 0 \\ -\infty; & \Re(a) < 0, \end{cases} \tag{35}$$

satisfies the requirement that $\hat{f}_m(\delta) = 0$ for $m \geq 0$.

In the same manner as before (see (15) and (16)), we can define an index of stability for using (31) in the forward direction:

$$\hat{\alpha}(k, m) := \left|\frac{\hat{\mathscr{J}}e_k(a, \delta, z)\, \hat{\mathscr{J}}e_m^{(h)}(a)}{\hat{\mathscr{J}}e_m(a, \delta, z)\, \hat{\mathscr{J}}e_k^{(h)}(a)}\right| = \frac{\hat{\rho}_m}{\hat{\rho}_k}, \tag{36}$$

where

$$\hat{\rho}_m := \left|\frac{\hat{\mathscr{J}}e_0(a, \delta, z)\, \hat{\mathscr{J}}e_m^{(h)}(a)}{\hat{\mathscr{J}}e_m(a, \delta, z)}\right|. \tag{37}$$

Now, the asymptotic behavior of $\hat{\rho}_m$, for large values of m, is obtained by substituting (34) and the result from Appendix A (see (96)) into (37):

$$\hat{\rho}_m^{(1)} \sim \begin{cases} e^{z\Re(a)}\sqrt{\pi m}\, |a(a^2+1)^m\, \hat{\mathscr{J}}e_0(a, \delta, z)|; & m \to \infty, z > 0, a \neq 0, \\ |\hat{\mathscr{J}}e_0(0, \delta, z)|; & m \to \infty, z > 0, a = 0. \end{cases} \tag{38}$$

This time, the condition in (19) is satisfied when $a \in D_a$, where

$$D_a := \{a : |a^2+1| \geq 1 \cap a \neq 0\}. \tag{39}$$

Therefore, forward recurrence using (31) is unstable when $a \in D_a$.

It is also interesting to look at the behavior of $\hat{\rho}_m$ for values of $m < z$. This behavior is obtained for the case $\Re(a) \geqslant 0$ by substituting (33) and the result from Appendix A (see (100)) into (37):

$$\hat{\rho}_m^{(2)} \sim \left| \hat{\mathscr{J}}e_0(a, \infty, z) \sqrt{\frac{z}{2}} \frac{\pi e^{az}(a^2+1)[z(a^2+1)/2]^m}{\Gamma(m+1/2)[a\cos(z - m\pi/2 - \pi/4) - \sin(z - m\pi/2 - \pi/4)]} \right|;$$

$$z > 0, \Re(a) \geqslant 0, \eta := \min(z, z|a \pm j|) \gg m \geqslant 0. \tag{40}$$

Since $\hat{\rho}_m^{(2)}$ becomes larger as m increases when $z|a^2+1|$ is large, forward recurrence will be unstable in the region where (40) holds.

Since $\hat{\rho}_m^{(1)}$ is monotone increasing as $m \to \infty$ for $a \in D_a$, it may be possible to use a Miller algorithm to calculate $\hat{\mathscr{J}}e_0(a, \delta, z)$ in this region of the a-plane. In the idealized case where infinite precision arithmetic is assumed, the relative error bound (see (24)) is given by

$$\left| \frac{\hat{\mathscr{J}}e_0^M(a, \delta, z) - \hat{\mathscr{J}}e_0(a, \delta, z)}{\hat{\mathscr{J}}e_0(a, \delta, z)} \right| = \left| \frac{\hat{\mathscr{J}}e_M(a, \delta, z)}{\hat{\mathscr{J}}e_M^{(h)}(a)\, \hat{\mathscr{J}}e_0(a, \delta, z)} \right| \tag{41}$$

$$= \frac{1}{\hat{\rho}_M^{(1)}}; \qquad M \to \infty, \tag{42}$$

where $\hat{\rho}_M^{(1)}$ is given in (38). Therefore, in this idealized case, $\hat{\mathscr{J}}e_0(a, \delta, z)$ can be computed to any number of significant digits by using Miller's algorithm when $a \in D_a$. Once again, it is advantageous to represent $\hat{\mathscr{J}}e_0(a, \delta, z)$ as a series. When $a \in D_a$, the desired series representation for $\hat{\mathscr{J}}e_0(a, \delta, z)$ is obtained by using (26), where $f_k(z)$ is replaced by $\hat{f}_k(z)$ and $Je_k^{(h)}(a)$ is replaced by $\hat{\mathscr{J}}e_k^{(h)}(a)$ (see (32) and (33)):

$$\hat{\mathscr{J}}e_m^M(a, \delta, z) = \frac{e^{-az}}{(a^2+1)\,\Gamma(m+1/2)} \sum_{k=m}^{M-1} \left[\frac{2}{a^2+1}\right]^{k-m} \frac{\Gamma(k+1/2)}{z^k}$$

$$\times [J_{k+1}(z) - aJ_k(z)]; \qquad z > 0, a \in D_a. \tag{43}$$

The integral of interest, $Je_0(a, z)$, is related to $\hat{\mathscr{J}}e_0(a, \delta, z)$ through the identity

$$Je_0(a, z) = -\hat{\mathscr{J}}e_0(a, \delta, 0) + \hat{\mathscr{J}}e_0(a, \delta, z), \tag{44}$$

where δ is defined in (35). The first integral, $\hat{\mathscr{J}}e_0(a, \delta, 0)$, is a special case of the integral in [13, (6.611.1)]. If $\Re(a) \geqslant 0$ and $a \neq \pm j$, then

$$\hat{\mathscr{J}}e_0(a, \infty, 0) = \int_\infty^0 e^{-at} J_0(t)\, dt = \frac{-1}{\sqrt{a^2+1}}. \tag{45}$$

On the other hand, when $\Re(a) < 0$, the change of variables $\tau = -t$ gives

$$\hat{\mathscr{J}}e_0(a, -\infty, 0) = \int_0^\infty e^{a\tau} J_0(\tau) = \frac{1}{\sqrt{a^2+1}}. \tag{46}$$

The square roots in (45) and (46) are defined as

$$\Re(\sqrt{a^2+1}) \geqslant 0. \tag{47}$$

A convergent factorial-Neumann series expansion is now obtained by combining the results in (43)–(47):

$$Je_0^M(a,z) = \frac{1}{\sqrt{a^2+1}} + \frac{e^{-az}}{\Gamma(1/2)(a^2+1)} \sum_{k=0}^{M-1} \left[\frac{2}{z(a^2+1)}\right]^k \times \Gamma\left(k+\frac{1}{2}\right)[J_{k+1}(z) - aJ_k(z)]; \qquad z>0,\ |a^2+1| \geqslant 1,\ a \neq 0, \tag{48}$$

where the proper branch cut for the square root is given by

$$\begin{aligned} \Re(\sqrt{a^2+1}) &\geqslant 0; \qquad \Re(a) \geqslant 0, \\ \Re(\sqrt{a^2+1}) &< 0; \qquad \Re(a) < 0. \end{aligned} \tag{49}$$

This branch cut is shown pictorially in Fig. 1.

When $|a^2+1| \geqslant 1$, but is not too large, the series expansion (48) will not yield an accurate approximation for $Je_0(a,z)$ until $M \gg z$. Therefore, for large values of z, a large number of terms may be required. Some insight into the behavior of this series is obtained by looking at the special case $|a^2+1| = 1$, where $a \neq 0$. If z is large, then the magnitude of the first few terms in the series will decrease rapidly due to the inverse powers in z. At some point, the behavior of the gamma function in the numerator will become dominant, and the terms will start to increase in magnitude. When k becomes much larger than z, the behavior of the Bessel functions dominates, and once again the terms will decrease in magnitude. Now, if $z|a^2+1|$ is large, what if we truncate the series when the magnitude of the terms

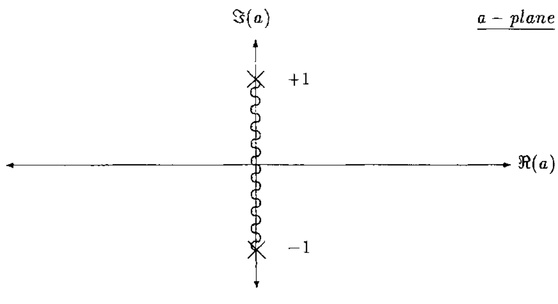

FIG. 1. Branch cut for the convergent series expansions.

reaches the first local minimum? This question can be answered by looking at the relative error bound (41) for $M \ll \eta = \min(z, z|a \pm j|)$. We find that

$$\left| \frac{\hat{\mathscr{J}}e_0^M(a, \infty, z) - \hat{\mathscr{J}}e_0(a, \infty, z)}{\hat{\mathscr{J}}e_0(a, \infty, z)} \right| = \frac{1}{\hat{\rho}_M^{(2)}}; \qquad \eta \gg M, z > 0, \Re(a) \geqslant 0, \tag{50}$$

where $\hat{\rho}_M^{(2)}$ is given in (40). Reference to (40) and (50) shows that for large values of $z|a^2+1|$ it may be possible to obtain an accurate approximation for $\hat{\mathscr{J}}e_0(a, \infty, z)$ by using the first few terms in (43). Therefore, the factorial-Neumann series (48) can be used as an asymptotic series for large values of $z|a^2+1|$, provided that $z > 0$ and $\Re(a) \geqslant 0$.

In Appendix A, we found that $\hat{\mathscr{J}}e_m(a, \infty, z)$ can be approximated by (100) for large values of η when $\Re(a) \geqslant 0$. We obtained this approximation by replacing the Bessel function in $\hat{\mathscr{J}}e_m(a, \infty, z)$ with its asymptotic expansion for large argument, and then we integrated the result. When z is a large positive number and $\Re(a) < 0$, the Bessel function in $\hat{\mathscr{J}}e_m(a, -\infty, z)$ cannot be replaced by its asymptotic expansion, since the integration variable now ranges between z and $-\infty$. If we rewrite $\hat{\mathscr{J}}e_m(a, -\infty, z)$ as

$$\hat{\mathscr{J}}e_m(a, -\infty, z) = \hat{\mathscr{J}}e_m(a, -\infty, z_0) + \hat{\mathscr{J}}e_m(a, z_0, z), \tag{51}$$

where $z < z_0$, then we can use (100) to show that

$$\hat{\mathscr{J}}e_m(a, -\infty, z) \sim C_m(a) - \sqrt{\frac{2}{\pi}} \frac{e^{-az}}{(a^2+1)\, z^{m+1/2}} \left[a \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right]; \qquad z > 0, \Re(a) \leqslant 0, \eta \gg m \geqslant 0, \tag{52}$$

where $C_m(a)$ is a function that is independent of z.

Using (33), (37), and (52), we find that

$$\hat{\rho}_m^{(2)} \sim |\hat{\mathscr{J}}e_0(a, -\infty, z)| \Bigg/ \left| \left\{ \frac{\Gamma(m+1/2)\, C_m(a)}{\Gamma(1/2)} - \sqrt{\frac{2}{z}} \frac{e^{-az}\Gamma(m+1/2)[a\cos(z - m\pi/2 - \pi/4) - \sin(z - m\pi/2 - \pi/4)]}{\pi(a^2+1)\, z^m} \right\} \left[\frac{2}{z(a^2+1)} \right]^m \right|;$$
$$z > 0, \Re(a) \leqslant 0, \eta \gg m \geqslant 0. \tag{53}$$

If we try to use (43) as an asymptotic expansion for $\hat{\mathscr{J}}e_0(a, -\infty, z)$ when $\Re(a) < 0$, then the relative error (see (50)) will be given by $1/\hat{\rho}_M^{(2)}$, where $\hat{\rho}_M^{(2)}$ is given in (53). Reference to (53) shows that because of the function $C_m(a)$, large values of $z|a^2+1|$ will not necessarily guarantee a small relative error. Therefore, (43) may not be used as an asymptotic expansion for $\hat{\mathscr{J}}e_0(a, -\infty, z)$ when $\Re(a) < 0$ and η is large.

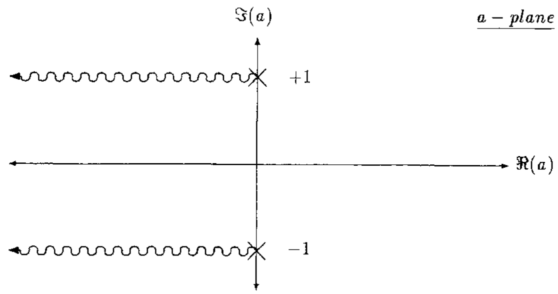

FIG. 2. Branch cut for the asymptotic series expansion.

However, (43) is an asymptotic expansion for some function; therefore, if we let (see (43))

$$Je_0^M(a, z) - G(a) \sim \frac{e^{-az}}{\Gamma(1/2)(a^2+1)} \sum_{k=0}^{M-1} \left[\frac{2}{z(a^2+1)}\right]^k \Gamma\left(k+\frac{1}{2}\right) [J_{k+1}(z) - aJ_k(z)], \quad (54)$$

then all we need to do is find the function $G(a)$. The integral, $Je_0(a, z)$, is continuous across the boundary $\Re(a) = 0$, therefore the asymptotic expansion for $Je_0(a, z)$ must also be continuous at this boundary. We previously found that (48) can be used as an asymptotic expansion for large $z|a^2+1|$ when $z > 0$ and $\Re(a) \geqslant 0$. Therefore, the function $G(a)$ can be determined by equating (48) with (54) at the boundary $\Re(a) = 0$. Doing this, we find that

$$G(a) = 1/\sqrt{a^2+1}, \quad (55)$$

where the branch cut for the square root must be defined as in Fig. 2. The branch cut is defined analytically by

$$\begin{aligned} &\Re(\sqrt{a^2+1}) < 0; \qquad a \in \{\Re(a) < 0\} \cap \{|\Im(a)| > 1\}, \\ &\Re(\sqrt{a^2+1}) \geqslant 0; \qquad \text{otherwise.} \end{aligned} \quad (56)$$

Finally, the desired asymptotic factorial-Neumann series expansion for $Je_0(a, z)$ is given by

$$\begin{aligned} Je_0^M(a, z) = {} & \frac{1}{\sqrt{a^2+1}} + \frac{e^{-az}}{\Gamma(1/2)(a^2+1)} \sum_{k=0}^{M-1} \left[\frac{2}{z(a^2+1)}\right]^k \Gamma\left(k+\frac{1}{2}\right) \\ & \times [J_{k+1}(z) - aJ_k(z)]; \qquad z > 0,\ z|a^2+1| \to \infty,\ a \in \mathbb{C}, \end{aligned} \quad (57)$$

where the proper branch cut for the square root is defined in (56). This branch cut was verified numerically (see Table I).

TABLE I

Typical Results for a Given Set of Inputs

Inputs				Use forward (F) or backward (B) recurrence	CPU time (ms)	No. of correct digits
z	a	SD	Use			
25.0	$0.0+j\,0.0$	4	(57)	F	4.67	5
		8	(29)	B	16.7	11
		8	D01AKF		96.0	14
		12	(58)	B	16.70	13
29.0	$0.2+j\,1.0$	4	(57)	F	4.00	6
		8	(29)	F	15.30	10
		8	D01AKF		104.0	14
		12	(29)	B	14.6	13
35.0	$0.0+j\,0.72$	4	(29)	F	16.7	5
		8	(29)	B	14.7	10
		8	D01AKF		102.0	13
		12	(29)	B	18.0	13
40.0	$-0.001+j\,0.6$	4	(57)	F	5.33	5
		8	(29)	B	20.0	11
		8	D01AKF		106.0	14
		12	(58)	B	23.3	14
40.0	$-0.001+j\,1.4$	4	(57)	F	5.33	6
		8	(57)	F	10.7	9
		8	D01AKF		320.0	14
		12	(58)	B	12.7	14
52.0	$-0.25+j\,1.0$	4	(57)	F	6.00	4
		8	(29)	F	28.0	10
		8	D01AKF		320.0	13
		12	(58)	B	15.3	13
100.0	$0.0-j\,0.9$	4	(57)	F	6.00	5
		8	(29)	F	20.7	10
		8	D01AKF		732.0	14
		12	(29)	F	24.7	13

It is interesting to compare the asymptotic expansion in (57) with the convergent series expansion (48). We find that the first term is not the same when $\Re(a)<0$ and $|\Im(a)|<1$. In fact, referring to (49) and (56) shows that they differ by a minus sign. This behavior is similar to the Stokes' phenomenon which occurs in asymptotic representations of special functions (see [17, Section 3.5]). There may also be a connection with the intrinsic cut appearing in partial fraction representations, which is discussed in [18].

5. Other Expansions for $Je_0(a, z)$

In this section, we look at some of the other expansions for $Je_0(a, z)$ that are given in the literature. A very useful Neumann series expansion is given by Agrest [7, (3.3) and (3.5)],

$$Je_0(a, z) = \frac{1}{\sqrt{a^2+1}} \left\{ 1 + e^{-az} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \varepsilon_k J_k(z)}{[\sqrt{a^2+1}+a]^k} \right\};$$

$$z > 0, a \in \mathbb{C}, a \neq \pm j, \tag{58}$$

where the branch cut for the square root is defined analytically in (49) and shown pictorially in Fig. 1, and ε_k is defined as

$$\varepsilon_k := \begin{cases} 1; & k=0, \\ 2; & k=1, 2, 3, \ldots. \end{cases} \tag{59}$$

This Neumann series expansion has good convergence properties for small to moderate values of z or large values of $|\sqrt{a^2+1}+a|$. Therefore, it will be used in conjunction with the two factorial-Neumann series expansions, which were derived in Sections 3 and 4, in an algorithm for the computation of $Je_0(a, z)$.

A Neumann series expansion for $Je_0(a, z)$ can also be found in an earlier paper by Maximon (see [9, (31′)]). By using the generating series [12, (9.1.41)] and some algebra, it can be shown that Maximon's expansion is equivalent to (58).

Other expansions, given in the literature, did not provide any significant computational advantages when they were compared with (29), (57), and (58); therefore, they were not included in the algorithm for the computation of $Je_0(a, z)$. However, some of these expansions will be briefly discussed below.

First, we will look at a convergent series expansion for $Je_0(a, z)$ that is given in the paper by Agrest and Rikenglaz [6, (6)]. This expansion is obtained by replacing the Bessel function in the integrand of $Je_0(a, z)$ by the power series expansion [12, (9.1.12)], and then integrating term-by-term. This procedure yields an expansion in terms of incomplete gamma functions, $\gamma(n+1, az)$, which can be expressed in the form

$$Je_0(a, z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 \, 4^k} g_{2k}(a, z), \tag{60}$$

where

$$g_n(a, z) := \frac{\gamma(n+1, az)}{a^{n+1}}; \; n \geq 0. \tag{61}$$

Incomplete gamma functions satisfy the recurrence relation [12, (6.5.22)], therefore $g_n(a, z)$ will satisfy

$$g_n(a, z) - \frac{a g_{n+1}(a, z)}{n+1} = \frac{z^{n+1} e^{-az}}{n+1}; \qquad n \geqslant 0. \tag{62}$$

Once again, the stability analysis presented in [14] can be applied to this recurrence relation. It shows that forward recurrence using (62) is unstable (see [15, Section 4] for more details). On the other hand, it can be shown that the sequence of functions, $\{g_n(a, z)\}$, can be computed using a backward recurrence algorithm. In [15, Section 4], the series expansion (60) is compared with the expansions in (29) and (58). When $|a^2+1| < 1$, it was found that (29) converges with fewer terms than (60). It was also found that round-off error is more of a problem when using (60) than when using (29) whenever z is large and $|a^2+1| < 1$. When $|a^2+1| > 1$, it can also be shown that (58) is better suited for computing $Je_0(a, z)$ than (60). We thus conclude that (60) does not offer any significant computational advantages when compared with (29) and (58).

The paper by Agrest and Rikenglaz (see [6, (4) and (5)]) also contains two asymptotic expansions for $Je_0(a, z)$ for large $z|a^2+1|$. Since $Je_0(a, z)$ must have a unique asymptotic expansion as $z|a^2+1| \to \infty$, the asymptotic expansions in [6] must be equivalent to (57). We found that the asymptotic factorial-Neumann series expansion (57) is better suited for computational purposes than either of the asymptotic expansions in [6].

In the paper (see [10, (3.2)]), Amos and Burgmeier give a second-order nonhomogeneous recurrence relation which can be used to obtain $Je_0(a, z)$. For the special case that we are interested in, this recurrence relation is equivalent to the recurrence relation that is given by Agrest in [7]. In that paper, Agrest shows that using this second-order recurrence relation is equivalent to summing the Neumann series (58).

6. An Algorithm for Computing $Je_0(a, z)$

In this section, we outline an algorithm which efficiently computes $Je_0(a, z)$ to a user defined number of significant digits (SD). We will use the three series expansions, (29), (57), and (58), to compute the ILHI, $Je_0(a, z)$, for $z > 0$ and $a \in \mathbb{C}$. For $z < 0$, we apply the identity (3).

As we have previously shown, these three expansions have very different properties. Therefore, we must determine which expansion to use for a given set of inputs, a, z, and SD. In all three of these expansions, we need to compute a sequence of Bessel functions $\{J_k(z)\}$. We use one of two different methods to compute the Bessel functions depending on the values of the inputs, a, z, and SD. In both of these methods, we will make use of the following recurrence relation (see [12, (9.1.27)]):

$$Z_{k-1}(z)+Z_{k+1}(z)=\frac{2k}{z}Z_k(z). \tag{63}$$

It is a well-known fact that $J_k(z)$ is the minimal solution of this recurrence relation, and $Y_k(z)$ is the dominant solution (see [14, 19]). Therefore, care must be taken when using (63) to obtain the sequence of Bessel functions.

The first method, which is the most efficient of the two methods and will be used whenever possible, involves the use of (63) in the forward direction. Since $J_k(z)$ is the minimal solution of the recurrence relation, forward recurrence will become unstable at some point. For values of $k<z$, $J_k(z)$ and $Y_k(z)$ both behave like damped trigonometric functions of z, but when $k>z$, these two functions behave very differently (see [12, (9.3.1)]):

$$J_\nu(z)\sim\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^{\nu}; \qquad \nu\to\infty, \tag{64}$$

$$Y_\nu(z)\sim-\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu}; \qquad \nu\to\infty. \tag{65}$$

Using the above equations, it can be shown that forward recurrence will become unstable when k becomes larger than z (see [14, 19]). Actually, it may be possible to compute a few Bessel functions of order greater than z, but the accuracy will start falling off rapidly with increasing values of k. Since we can only use forward recurrence to obtain $\{J_k(z)\}$ for $k\leqslant z$, this method will only be useful when z is large. When this is the case, we can use Hankel's asymptotic expansion to compute the two starting functions, $J_0(z)$ and $J_1(z)$, and then we can use (63) in the forward direction to compute $\{J_k(z)\}$ for $k\leqslant z$. We determined using numerical tests that Hankel's asymptotic expansion can be used to approximate the starting functions to SD significant digits if

$$z>SD+4. \tag{66}$$

When we cannot use forward recurrence to compute the sequence of Bessel functions, then we must resort to a backward recurrence algorithm. We will make use of a backward recurrence algorithm which is based upon a combination of the algorithms due to J. C. P. Miller and F. W. J. Olver (see [20]). We chose this algorithm because it automatically computes the sequence of Bessel functions to a user defined number of significant digits. Background information on the Miller and Olver algorithms can be found in [14, 19, 21].

In order to compute $Je_0(a,z)$ to SD significant digits using (29) or (58), we need accurate approximations for all the Bessel functions, $J_k(z)$, that are larger in magnitude than the test value, T, where

$$T:=\tfrac{1}{2}\times10^{-SD}|J_0(z)|. \tag{67}$$

The test value, T, can be approximated by

$$T \approx \begin{cases} \frac{1}{2} \times 10^{-SD}; & 0 \leqslant z \leqslant 1 \\ \frac{1 \times 10^{-SD}}{\sqrt{2\pi z}}; & z > 1. \end{cases} \tag{68}$$

The algorithm in [20] provides a very efficient method for accomplishing this.

First, a sequence $p_M, p_{M+1}, \ldots$ is computed by using (63) in the forward direction starting with $p_M = 0$ and $p_{M+1} = 1$, where $M :=$ int(z) (M is the largest integer less than or equal to z). The sequence of p_m's will be nondecreasing in absolute value since the dominant solution to the recurrence relation, $Y_m(z)$, is nondecreasing for $m > M$. Forward recurrence continues until the error condition,

$$\frac{1}{|p_{N+1}|} < T, \tag{69}$$

is satisfied (see [21, (4.12)]), where T is given in (68).

Once the error condition in (69) is satisfied, backward recurrence on (63), beginning with $Z_N = 0$ and $Z_{N-1} = 1/p_N$, gives the sequence $Z_N, Z_{N-1}, \ldots, Z_0$. Now, the desired solutions, $J_k(z)$, are determined by applying the normalization condition [12, (9.1.46)]:

$$J_k(z) = \frac{Z_k}{Z_0 + 2Z_2 + 2Z_4 + \cdots + 2Z_{N-1}}. \tag{70}$$

The next step is to determine which expansion to use to compute $Je_0(a, z)$ for a given set of inputs, a, z, and SD. For large values of z and $z|a^2+1|$, we would like to use the asymptotic factorial-Neumann series expansion (57). Since z is large, we will use forward recurrence to compute the sequence of Bessel functions. Therefore, we must truncate the series at some k, where $k \leqslant z$. This means that the Bessel functions in (57) will have sinusoidal behavior. Reference to (57) shows that we will obtain the best approximation for $Je_0(a, z)$ when the factor,

$$\left[\frac{2}{z|a^2+1|}\right]^k \Gamma\left(k+\frac{1}{2}\right), \tag{71}$$

reaches the first minimum. Using the results which are derived in Appendix B (see (106)), we find that (71) reaches a minimum when

$$k = k_{\max} := \frac{z|a^2+1|-1}{2}. \tag{72}$$

If $k_{\max} > z$, then the accuracy is limited by the number of computed Bessel functions. For this case, we will set

$$k_{\max} = z. \tag{73}$$

We can use the series expansion (57) to obtain an approximation for $Je_0(a, z)$ that is accurate to SD significant digits if the kth term, where $k = k_{\max}$, is small enough. Therefore, the following inequality must hold:

$$\frac{1}{2} \times 10^{-SD} |e^{az} Je_0(a, z)| > \left| \frac{\Gamma(k_{\max} + 1/2)}{\Gamma(1/2)(a^2+1)} \left[\frac{2}{z(a^2+1)} \right]^{k_{\max}} [J_{k_{\max}+1}(z) - aJ_{k_{\max}}(z)] \right|. \tag{74}$$

For large values of $\eta = \min(z, z|a \pm j|)$, we can obtain an approximation for (74) by using [12, (9.2.1) and (6.1.37)]:

$$\frac{1}{2} \times 10^{-SD} |e^{az} Je_0(a, z)| > \frac{2}{\sqrt{\pi e z}\, |a^2+1|} \left[\frac{2k_{\max} + 1}{ze|a^2+1|} \right]^{k_{\max}} \max(1, |a|). \tag{75}$$

When $z > k_{\max}$, (75) can be further simplified by substituting (72) into (75):

$$\frac{1}{2} \times 10^{-SD} |e^{az} Je_0(a, z)| > \frac{2}{|a^2+1| \sqrt{\pi z}}\, e^{-z(|a^2+1|/2)} \max(1, |a|). \tag{76}$$

An approximation for $|e^{az} Je_0(a, z)|$ can be obtained by using the previous result (21):

$$|e^{az} Je_0(a, z)| \sim \left| \frac{e^{az}}{\sqrt{a^2+1}} - \sqrt{\frac{2}{\pi z}} \frac{[a \cos(z - \pi/4) - \sin(z - \pi/4)]}{(a^2+1)} \right|; \quad \min(z, z|a \pm j|) \gg 0. \tag{77}$$

Reference to (75)–(77) shows that the asymptotic factorial-Neumann series expansion (57) is most useful when $z|a^2+1|$ is large. We use this expansion to compute $Je_0(a, z)$ when (75) holds and $z > SD + 4$.

When $z > SD + 4$, but (75) is not satisfied, we still prefer to use forward recurrence to compute the sequence of Bessel functions, but now we would like to use the convergent factorial-Neumann series expansion (29) to compute $Je_0(a, z)$. Once again, we must first determine whether this expansion will provide the desired accuracy. Since we are using forward recurrence to obtain $\{J_k(z)\}$, the series (29) must be truncated at some $k \leqslant k_{\text{int}} := \text{int}(z)$. Therefore, we can use the series expansion (29) provided that

$$\frac{1}{2} \times 10^{-SD} |e^{az} Je_0(a, z)| > \left| \Gamma\left(\frac{3}{2}\right) z \left[\frac{z(a^2+1)}{2} \right]^{k_{\text{int}}} \frac{[J_{k_{\text{int}}}(z) + aJ_{k_{\text{int}}+1}(z)]}{\Gamma(k_{\text{int}} + 3/2)} \right|. \tag{78}$$

Once again, we approximate (78) by applying [12, (9.2.1) and (6.1.37)]:

$$\frac{1}{2} \times 10^{-SD} |e^{az} Je_0(a, z)| > \sqrt{\frac{z}{\pi}} \frac{e^{3/2}}{(2k_{\text{int}} + 3)} \left[\frac{ez|a^2+1|}{2k_{\text{int}} + 3} \right]^{k_{\text{int}}} \max(1, |a|). \tag{79}$$

When $z|a^2+1| \geqslant 2$, we can use the approximation for $|e^{az}Je_0(a, z)|$ which is given in (77). On the other hand, when $z|a^2+1|<2$, we can obtain an adequate approximation for $|e^{az}Je_0(a, z)|$ by keeping the first term in (29), yielding

$$|e^{az}Je_0(a, z)| \approx z|J_0(z)+aJ_1(z)|$$
$$\sim \sqrt{\frac{2z}{\pi}}\left|\cos\left(z-\frac{\pi}{4}\right)+a\cos\left(z-\frac{3\pi}{4}\right)\right|; \qquad z|a^2+1|<2, \tag{80}$$

where the asymptotic expansion [12, (9.2.1)] has been applied. We use forward recurrence to compute the Bessel functions and then use the convergent factorial-Neumann series expansion (29) when $z > SD+4$ and (79) holds.

If this expansion will not work, then we must compute the sequence of Bessel functions by using backward recurrence, and then use one of the expansions given in (29) or (58). When $z|a^2+1| \leqslant 2$, we use the convergent factorial-Neumann series expansion (29), since it converges faster than the Neumann series expansion (58).

When $z|a^2+1|>2$ and $|a^2+1| \leqslant 1$, the convergent factorial-Neumann series expansion will still converge faster than the Neumann series expansion, but now we need to worry about round-off errors. In Section 3, we found that the $1/\rho_k^{(2)}$ term in (25) may become large and cause round-off error problems when $z|a^2+1|$ is large. Since we want to obtain an approximation to $Je_0(a, z)$ which is accurate to SD significant digits, reference to (22) and (25) shows that

$$\frac{1}{2}\times 10^{-SD}|Je_0(a, z)| > \frac{\varepsilon|Je_0(a, z)|}{\rho_{k_{\max}}^{(2)}}$$
$$=\varepsilon\left|\frac{1}{\sqrt{a^2+1}}-\sqrt{\frac{2}{z}}\,e^{-az}\left|\frac{-z(a^2+1)}{2}\right|^{k_{\max}}\right.$$
$$\left.\times\frac{[a\cos(z+\pi k_{\max}/2-\pi/4)-\sin(z+\pi k_{\max}/2-\pi/4)]}{\Gamma(k_{\max}+1/2)(a^2+1)}\right|, \tag{81}$$

where $k_{\max}$ is chosen to maximize the second term on the right-hand side of (81). Applying the result from Appendix B (see (106)) we find that (72) still holds.

Round-off error will not be a problem if the second term in (81) is smaller than the first term. If this is the case, then (29) can be used to compute $Je_0(a, z)$. On the other hand, when the second term is larger than the first term, we must take round-off errors into account. Application of [12, (6.1.37)] and (72) to (81) yields

$$\varepsilon<\frac{1}{2}\times 10^{-SD}|Je_0(a, z)(a^2+1)|\,\frac{\sqrt{\pi z}\,e^{z[\Re(a)-|a^2+1|/2]}}{\max(1, |a|)}, \tag{82}$$

where we have dropped the first term on the right-hand side of (81). This inequality gives us an estimate for the maximum tolerable error, ε, that can exist if (29) is to be used when $z|a^2+1|$ is large and $|a^2+1| \leqslant 1$. We can define

$$SDN := SD - \log_{10}\left(\frac{|a^2+1|\,\sqrt{\pi z}\,e^{z[\Re(a) - |a^2+1|/2]}}{\max(1, |a|)}\right), \tag{83}$$

where SDN is the number of significant digits required in all operations. If the accuracy of the computer is less than SDN, then (29) cannot be used to compute $Je_0(a, z)$. If (29) can be used, then in order to obtain SD significant digits in $Je_0(a, z)$, we must use SDN instead of SD when we calculate the sequence of Bessel functions (see (68)).

Finally, if the parameters a, z, and SD are such that none of the previously mentioned methods can be used, then we will use backward recurrence to compute the

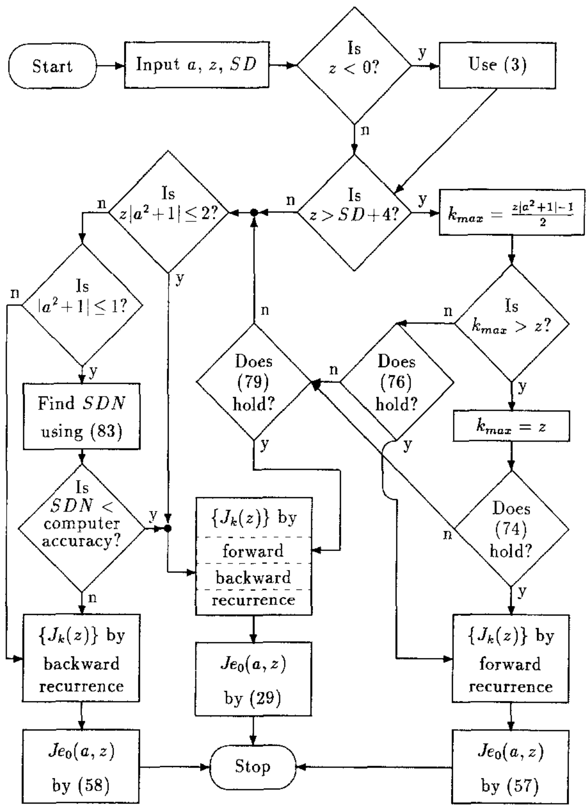

FIG. 3. Flow chart for the computation of $Je_0(a, z)$.

sequence of Bessel functions, and the Neumann series expansion (58) will be used to calculate $Je_0(a, z)$.

An algorithm which is structured as outlined in this section (see the flow chart in Fig. 3) can be used to determine which expansion to use to compute $Je_0(a, z)$. We will name this algorithm TJE0. A listing of the Fortran source code for TJE0 is given in Appendix F of [15].

TABLE II

Typical Results for a Given Set of Inputs

Inputs				Use forward (F) or backward (B) recurrence	CPU time (ms)	No. of correct digits
z	a	SD	Use			
0.01	$0.1 + j\,5.0$	4	(29)	B	2.00	10
		8	(29)	B	3.33	14
		8	D01AKF		98.0	14
		12	(29)	B	3.33	14
1.0	$0.0 + j\,1.0$	4	(29)	B	2.67	6
		8	(29)	B	3.33	11
		8	D01AKF		94.0	14
		12	(29)	B	3.33	14
1.0	$-20.0 + j\,20.0$	4	(58)	B	2.67	6
		8	(58)	B	4.00	12
		8	D01AKF		98.0	14
		12	(58)	B	4.67	14
1.0	$20.0 + j\,1000.0$	4	(58)	B	0.667	8
		8	(58)	B	0.667	8
		8	D01AKF		3048.0	14
		12	(58)	B	3.33	13
10.0	$-1.0 + j\,20.0$	4	(57)	F	4.00	6
		8	(58)	B	5.33	11
		8	D01AKF		708.0	13
		12	(58)	B	5.33	13
100.0	$0.1 + j\,5.0$	4	(57)	F	4.00	14
		8	(57)	F	4.00	14
		8	D01AKF		1976.0	14
		12	(57)	F	5.33	14
1000.0	$-0.01 + j\,1.0$	4	(29)	F	20.0	6
		8	(29)	F	24.0	10
		8	D01AKF		7000.0	13
		12	(29)	F	26.7	12

7. Results

In this section, we use TJE0 to compute $Je_0(a, z)$ for some typical values of a, z, and SD. For comparison purposes, we also use an adaptive quadrature routine (D01AKF) from the NAG library [11]. D01AFK was chosen because it is well suited for oscillating, non-singular integrands. The relative error bound,

$$EPSREL = \tfrac{1}{2} \times 10^{-SD}, \tag{84}$$

will be used with this routine. The Bessel function in the integrand of $Je_0(a, z)$ is computed using another NAG routine (S17AEF). The tests were carried out on a Hewlett Packard 9000 series 300 workstation.

Table I and Table II contain the results for some typical values of the inputs, a, z, and SD. In the fourth column, we indicate which method was used to compute $Je_0(a, z)$. When the quadrature routine was used, D01AKF is listed. On the other hand, when TJE0 is used, the algorithm automatically chooses which of the expansions, (29), (57), or (58), to use. Therefore, the expansion which was chosen for a given set of inputs is listed in the fourth column. The algorithm TJE0 also chooses whether to use forward or backward recurrence to compute the sequence of Bessel functions. This is listed in the fifth column. These two columns are included in the table to demonstrate how TJE0 automatically picks which method to use for a given set of inputs. Even though the method that is chosen is not guaranteed to be the most efficient method, it usually is.

The sixth column shows the amount of CPU time which was required to compute $Je_0(a, z)$ for a given set of inputs. This column shows that TJE0 offers a very efficient way to compute $Je_0(a, z)$ when compared with D01AKF. It also shows how efficient the different series expansions are for a given set of inputs.

The last column contains the number of correct digits in the final result. While neither routine will guarantee the requested number of significant digits, this column demonstrates that the request is usually satisfied.

In conclusion, we find that TJE0 offers a very accurate and efficient way to compute $Je_0(a, z)$ for a given set of inputs, a, z, and SD. The algorithm accomplishes this by choosing between the three series expansions, (29), (57), or (58), where the required sequence of Bessel functions is computed using either forward or backward recurrence.

In [4, 5], the inner angular integral of a two-dimensional Sommerfeld integral was written in terms of a finite number of ILHIs, $Je_0(a, z)$. In these papers, it was found that a modified version of TJE0 provides an efficient way to compute the required ILHIs.

Appendix A: Asymptotic Approximations for Some Integrals

In this appendix, we derive asymptotic approximations for the integrals we encounter while performing the error analysis in Sections 3 and 4.

A.1. *Asymptotic Behavior of $Je_n(a, z)$ for Large n*

The function of interest is given in (4). If $n \gg z$, then an asymptotic expansion for this integral can be obtained by replacing the Bessel function by its asymptotic expansion (64)

$$Je_n(a, z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{e}{2n}\right)^n \int_0^z e^{-at} t^{2n} dt; \qquad n \gg z. \tag{85}$$

The integral on the right-hand side of (85) is an incomplete gamma function (see [12, (6.5.2) and (6.5.4)]) which can be expanded in terms of a series by using [12, (6.5.29)]:

$$Je_n(a, z) \sim \frac{ze^{-az}}{\sqrt{2\pi n}} \left(\frac{z^2 e}{2n}\right)^n \Gamma(2n+1) \sum_{k=0}^{\infty} \frac{(az)^k}{\Gamma(k+2n+2)}; \qquad n \gg z. \tag{86}$$

Now, the desired asymptotic behavior is obtained by only keeping the first term in this series:

$$Je_n(a, z) \sim \frac{ze^{-az}}{\sqrt{2\pi n}} \left(\frac{z^2 e}{2n}\right)^n \frac{1}{2n+1};$$

$$n \gg \kappa := \mathrm{int}(\max(z, |az|)), \; z > 0, \; a \in \mathbb{C}. \tag{87}$$

A.2. *Asymptotic Behavior of $\hat{\mathcal{J}}e_m(a, \delta, z)$ for Large m*

We are interested in finding the asymptotic behavior of (30), where δ is defined in (35). This integral is first divided into two pieces,

$$\hat{\mathcal{J}}e_m(a, \delta, z) = \hat{\mathcal{J}}e_m(a, \delta, 0) + \hat{\mathcal{J}}e_m(a, 0, z), \tag{88}$$

and then the asymptotic behavior of each piece is obtained separately. The results are then added together to obtain the asymptotic behavior of the original integral (30).

The asymptotic behavior of $\hat{\mathcal{J}}e_m(a, 0, z)$ is obtained by replacing the Bessel function with its asymptotic expansion (64):

$$\hat{\mathcal{J}}e_m(a, 0, z) \sim \begin{cases} \dfrac{1}{\sqrt{2\pi m}} \left(\dfrac{e}{2m}\right)^m \dfrac{(1 - e^{-az})}{a}; & m \gg z > 0, \; a \neq 0, \\[2ex] \dfrac{z}{\sqrt{2\pi m}} \left(\dfrac{e}{2m}\right)^m; & m \gg z > 0, \; a = 0. \end{cases} \tag{89}$$

The other integral, $\hat{\mathcal{J}}e_m(a, \delta, 0)$, will exhibit a different asymptotic behavior for the two cases of δ given in (35). When $\Re(a) \geqslant 0$ and $a \neq 0$, $\hat{\mathscr{I}e}_m(a, \infty, 0)$ can be expressed as a hypergeometric function by using [13, (6.621.1)]:

$$\hat{\mathscr{I}e}_m(a, \infty, 0) = \frac{-1}{a 2^m \Gamma(m+1)} F\left(\frac{1}{2}, 1; m+1; \frac{-1}{a^2}\right). \tag{90}$$

The asymptotic behavior is now obtained by only keeping the first term in the series expansion for the hypergeometric function [12, (15.1.1)]:

$$\hat{\mathscr{I}e}_m(a, \infty, 0) \sim \frac{-1}{a 2^m \Gamma(m+1)}; \qquad m \gg \frac{1}{|a^2|}, \Re(a) \geqslant 0, a \neq 0. \tag{91}$$

On the other hand, when $a = 0$, the integral is known in closed form (see [13, (6.561.14)]):

$$\hat{\mathscr{I}e}_m(0, \infty, 0) = \frac{-\Gamma(1/2)}{2^m \Gamma(m+1/2)}. \tag{92}$$

Now, the asymptotic behavior of $\hat{\mathscr{I}e}_m(a, \infty, 0)$ can be expressed in a form similar to (89) by applying Stirling's formula and [12, (4.1.17)]:

$$\hat{\mathscr{I}e}_m(a, \infty, 0) \sim \begin{cases} \dfrac{-1}{a\sqrt{2\pi m}} \left(\dfrac{e}{2m}\right)^m; & m \to \infty, a \neq 0, \Re(a) \geqslant 0, \\ \dfrac{-1}{\sqrt{2}} \left(\dfrac{e}{2m}\right)^m; & m \to \infty, a = 0. \end{cases} \tag{93}$$

The asymptotic behavior of $\hat{\mathscr{I}e}_m(a, -\infty, 0)$ for the case $\Re(a) < 0$ is obtained by making the changes of variables, $\tau = -t$ and $b = -a$, in (30). This yields

$$\hat{\mathscr{I}e}_m(a, -\infty, 0) = -\hat{\mathscr{I}e}_m(b, \infty, 0), \tag{94}$$

where [12, (9.1.35)] was applied to the Bessel function. Now, the previous result (93) can be applied, since $\Re(b) > 0$:

$$\hat{\mathscr{I}e}_m(a, -\infty, 0) \sim \frac{1}{b\sqrt{2\pi m}} \left(\frac{e}{2m}\right)^m = \frac{-1}{a\sqrt{2\pi m}} \left(\frac{e}{2m}\right)^m; \qquad m \to \infty, \Re(a) < 0. \tag{95}$$

Finally, the asymptotic behavior of $\hat{\mathscr{I}e}_m(a, \delta, z)$ is obtained by combining the results given in (88), (89), (93), and (95):

$$\hat{\mathscr{I}e}_m(a, \delta, z) \sim \begin{cases} \dfrac{-e^{-az}}{a\sqrt{2\pi m}} \left(\dfrac{e}{2m}\right)^m; & m \to \infty, z > 0, a \neq 0, \\ \dfrac{-1}{\sqrt{2}} \left(\dfrac{e}{2m}\right)^m; & m \to \infty, z > 0, a = 0. \end{cases} \tag{96}$$

A.3. *Asymptotic Behavior of* $\hat{\mathscr{J}}e_m(a, \infty, z)$ *for Large* z

We are interested in finding the asymptotic behavior of (30) for large values of z when $\Re(a) \geqslant 0$. When $z \gg m$, the Bessel function in (30) can be replaced by its principle asymptotic form [12, (9.2.1)]:

$$\hat{\mathscr{J}}e_m(a, \infty, z) \sim \sqrt{\frac{2}{\pi}} \int_{\infty}^{z} e^{-at} t^{-m-1/2} \cos\left(t - \frac{m\pi}{2} - \frac{\pi}{4}\right) dt;$$
$$z \gg m \geqslant 0,\ \Re(a) \geqslant 0. \tag{97}$$

The analysis is simplified by initially assuming that a is a real variable and then later extending the results to complex values of a by using analytic continuation. This assumption allows us to rewrite (97) as

$$\hat{\mathscr{J}}e_m(a, \infty, z) \sim \sqrt{\frac{2}{\pi}}\, \Re \left\{ e^{\pm j(\pi(2m+1)/4)} \int_{\infty}^{z} e^{-t(a \pm j)} t^{-m-1/2} dt \right\};$$
$$a \geqslant 0,\ z \gg m \geqslant 0. \tag{98}$$

Making the change of variables, $\tau = t(a \pm j)$, now enables us to put this integral into the form of an incomplete gamma function (see [12, (6.5.3)]):

$$\hat{\mathscr{J}}e_m(a, \infty, z) \sim -\sqrt{\frac{2}{\pi}}\, \Re \left\{ e^{\pm j(\pi(2m+1)/4)} (a \pm j)^{m-1/2}\, \Gamma\left(\frac{1}{2} - m,\ z(a \pm j)\right) \right\};$$
$$a \geqslant 0,\ z \gg m \geqslant 0. \tag{99}$$

The desired asymptotic expansion is obtained by replacing the incomplete gamma function by the first term of [12, (6.5.32)]:

$$\hat{\mathscr{J}}e_m(a, \infty, z) \sim -\sqrt{\frac{2}{\pi}} \frac{e^{-az}}{(a^2+1)\, z^{m+1/2}}$$
$$\times \left[a \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) - \sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) \right];$$
$$z > 0,\ a \geqslant 0,\ \eta := \min(z, z|a \pm j|) \gg m \geqslant 0. \tag{100}$$

Now, by using analytic continuation, (100) can be extended to hold for $z > 0$, $\Re(a) \geqslant 0$, where $a \neq \pm j$.

With the added restriction, $\Re(a) \neq 0$, $\hat{\mathscr{J}}e_m(a, \infty, z)$ will be defined for all positive and negative values of m. Using Eq. (5), (30), and [12, (9.1.5)], we find that

$$\mathscr{J}e_n(a, \infty, z) = (-1)^n\, \hat{\mathscr{J}}e_{-n}(a, \infty, z); \qquad z > 0,\ \Re(a) > 0. \tag{101}$$

Therefore, the asymptotic expansion for $\mathcal{J}e_n(a, \infty, z)$ is found using the results in (100):

$$\mathcal{J}e_n(a, \infty, z) \sim (-1)^{n+1} \sqrt{\frac{2}{\pi}} \frac{e^{-az} z^{n-1/2}}{(a^2+1)} \left[a \cos\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right) - \sin\left(z + \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]; \qquad z > 0, \Re(a) > 0, \eta \gg n \geqslant 0. \tag{102}$$

APPENDIX B: Finding the Local Minimum of P_l

In this appendix, we determine what value of l minimizes P_l, where

$$P_l := \left(\frac{2}{z|a^2+1|}\right)^l \Gamma(l); \qquad l > 0. \tag{103}$$

The gamma function in (103) can be approximated by the first term in Stirling's formula for moderate to large values of l:

$$P_l \approx \sqrt{\frac{2\pi}{l}} \left(\frac{2l}{ze|a^2+1|}\right)^l; \qquad l \gg 0. \tag{104}$$

Now, the value of l which minimizes P_l can be found by differentiating (104) and setting the result to zero:

$$0 = \frac{\partial P_l}{\partial l} = P_l \left[1 + \ln\left(\frac{2l}{ez|a^2+1|}\right) - \frac{1}{2l}\right]; \qquad l \gg 0. \tag{105}$$

Solving this equation for l yields

$$l = \frac{z|a^2+1|\, e^{1/2l}}{2} \approx \frac{z|a^2+1|}{2}; \qquad l \gg 0. \tag{106}$$

References

1. M. M. Agrest and M. Z. Maksimov, *Theory of Incomplete Cylindrical Functions and Their Applications* (Springer-Verlag, Berlin, 1971).
2. P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University Press, Princeton, NJ, 1986), pp. 458.
3. E. F. Kuester and D. C. Chang, Scientific Report No. 43, Electromagnetics Laboratory, Department of Electrical Engineering, University of Colorado, Boulder, Colorado 80309, 1979. (unpublished)
4. S. L. Dvorak and E. F. Kuester, "Fast Numerical Computation of Two-Dimensional Sommerfeld Integrals Using Incomplete Lipschitz-Hankel Integrals," National Radio Science Meeting, URSI Abstracts, Boulder, Colorado, January 1987, p. 72.

5. S. L. Dvorak and E. F. Kuester, Scientific Report No. 94, Electromagnetics Laboratory, Department of Electrical Engineering, University of Colorado, Boulder, Colorado 80309, 1989. (unpublished)
6. M. M. Agrest and M. M. Rikenglaz, *Zh. Vychisl. Mat. Mat. Fiz.* **7**, 1370 (1967) [Russian]; *USSR Comput. Math. Math. Phys.* **7**, No. 6, 206 (1967).
7. M. M. Agrest, *Zh. Vychisl. Mat. Mat. Fiz.* **11**, 1127 (1971) [Russian]; *USSR Comput. Math. Math. Phys.* **11**, No. 5, 40 (1971).
8. M. M. Agrest, *Zh. Vychisl. Mat. Mat. Fiz.* **18**, 10 (1978) [Russian]; *USSR Comput. Math. Math. Phys.* **18**, No. 1, 8 (1978).
9. L. C. Maximon, *Proc. Amer. Math. Soc.* **7**, 1054 (1956).
10. D. E. Amos and J. W. Burgmeier, *SIAM Rev.* **15**, 335 (1973).
11. *The NAG Mark 12 Manual* (Numer. Algorithms Group, Oxford, 1987).
12. M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U. S. Govt. Printing Office, Washington, DC, 1972).
13. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, Orlando, 1980).
14. J. Wimp, *Computation with Recurrence Relations* (Pitman, Boston, 1984).
15. S. L. Dvorak and E. F. Kuester, Scientific Report No. 89, Electromagnetics Laboratory, Department of Electrical Engineering, University of Colorado, Boulder, Colorado 80309, 1987 (unpublished).
16. N. Nielsen, *Handbuch der Theorie der Cylinderfunktionen* (Druck Teubner, Leipzig, 1904), p. 261.
17. A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956).
18. M. V. Cerrillo, "The "Cliff" Method of Approximate Integration and the Radiation from a Dipole over a Finite Conductive Ground," in *The McGill Symposium on Microwave Optics, Part II: Diffraction and Scattering, Bedford, MA*, Astia Document No. AD 211500, edited by B. S. Karasik, Bedford, Mass., April 1959, p. 172.
19. W. Gautschi, *SIAM Rev.* **9**, 24 (1967).
20. F. W. J. Olver and D. J. Sookne, *Math. Comput.* **26**, 941 (1972).
21. F. W. J. Olver, *J. Res. Nat. Bur. Stand. Sect. B* **71**, 111 (1967).